

CENTRALIZING TRACES AND LIE TRIPLE ISOMORPHISMS ON TRIANGULAR ALGEBRAS

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ABSTRACT. Let \mathcal{T} be a triangular algebra over a commutative ring \mathcal{R} and $\mathcal{Z}(\mathcal{T})$ be the center of \mathcal{T} . Suppose that $q: \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ is an \mathcal{R} -bilinear mapping and that $\mathfrak{T}_q: \mathcal{T} \rightarrow \mathcal{T}$ is a trace of q . We describe the form of \mathfrak{T}_q satisfying the condition $[\mathfrak{T}_q(T), T] \in \mathcal{Z}(\mathcal{T})$ for all $T \in \mathcal{T}$. The question of when \mathfrak{T}_q has the proper form will be addressed. Using the aforementioned trace function, we establish sufficient conditions for each Lie triple isomorphism on \mathcal{T} to be almost standard. As applications we characterize Lie triple isomorphisms of triangular matrix algebras and nest algebras. Some further research topics related to current work are proposed at the end of this article.

1. INTRODUCTION

Let \mathcal{R} be a commutative ring with identity, \mathcal{A} be a unital algebra over \mathcal{R} and $\mathcal{Z}(\mathcal{A})$ be the center of \mathcal{A} . Let us denote the commutator or the Lie product of the elements $a, b \in \mathcal{A}$ by $[a, b] = ab - ba$. Recall that an \mathcal{R} -linear mapping $f: \mathcal{A} \rightarrow \mathcal{A}$ is said to be *semi-centralizing* if either $[f(a), a] \in \mathcal{Z}(\mathcal{A})$ or $f(a)a + af(a) \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. Further, the mapping f is said to be *centralizing* if $[f(a), a] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. The mapping f is said to be *skew-centralizing* if $f(a)a + af(a) \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. In particular, the mapping f is said to be *commuting* if $[f(a), a] = 0$ for all $a \in \mathcal{A}$. The mapping f is said to be *skew-commuting* if $f(a)a + af(a) = 0$ for all $a \in \mathcal{A}$. When we investigate the above-mentioned mappings, the principal task is to describe their forms. This is demonstrated by various works, see [8, 10, 11, 12, 13, 15, 20, 21, 28, 29, 33, 36, 41, 44, 45, 49, 50]. We encourage the reader to read the well-written survey paper [13], in which the author presented the development of the theory of semi-centralizing mappings and their applications in details.

Let \mathcal{R} be a commutative ring with identity, \mathcal{A} be a unital algebra over \mathcal{R} and $\mathcal{Z}(\mathcal{A})$ be the center of \mathcal{A} . Recall that an \mathcal{R} -linear mapping $f: \mathcal{A} \rightarrow \mathcal{A}$ is said to be *centralizing* if $[f(a), a] \in \mathcal{Z}(\mathcal{A})$ for all $a \in \mathcal{A}$. Let n be a positive integer and $q: \mathcal{A}^n \rightarrow \mathcal{A}$ be an n -linear mapping. The mapping $\mathfrak{T}_q: \mathcal{A} \rightarrow \mathcal{A}$ defined by $\mathfrak{T}_q(a) = q(a, a, \dots, a)$ is called a *trace* of q . We say that a centralizing trace \mathfrak{T}_q is *proper* if it can be written as

$$\mathfrak{T}_q(a) = za^n + \mu_1(a)a^{n-1} + \dots + \mu_{n-1}(a)a + \mu_n(a)$$

2000 *Mathematics Subject Classification.* 47L35, 15A78, 16W25.

Key words and phrases. Centralizing trace, Lie triple isomorphism, commuting trace, triangular algebra, nest algebra.

The work of the second author is supported by the Mathematical Tianyuan Fundamental of NSFC (Grant No. 11226068).

for all $a \in \mathcal{A}$, where $z \in \mathcal{Z}(\mathcal{A})$ and μ_i ($1 \leq i \leq n$) is a mapping from \mathcal{A} into $\mathcal{Z}(\mathcal{A})$ and every μ_i ($1 \leq i \leq n$) is in fact a trace of an i -linear mapping \mathbf{q}_i from \mathcal{A}^i into $\mathcal{Z}(\mathcal{A})$. Let $n = 1$ and $\mathbf{f}: \mathcal{A} \rightarrow \mathcal{A}$ be an \mathcal{R} -linear mapping. In this case, an arbitrary trace $\mathfrak{T}_{\mathbf{f}}$ of \mathbf{f} exactly equals to itself. Moreover, if a centralizing trace $\mathfrak{T}_{\mathbf{f}}$ of \mathbf{f} is proper, then it has the form

$$\mathfrak{T}_{\mathbf{f}}(a) \equiv za \pmod{\mathcal{Z}(\mathcal{A})}, \quad \forall a \in \mathcal{A},$$

where $z \in \mathcal{Z}(\mathcal{A})$. Let us see the case of $n = 2$. Suppose that $\mathbf{g}: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ is an \mathcal{R} -bilinear mapping. If a centralizing trace $\mathfrak{T}_{\mathbf{g}}$ of \mathbf{g} is proper, then it is of the form

$$\mathfrak{T}_{\mathbf{g}}(a) \equiv za^2 + \mu(a)a \pmod{\mathcal{Z}(\mathcal{A})}, \quad \forall a \in \mathcal{A},$$

where $z \in \mathcal{Z}(\mathcal{A})$ and μ is an \mathcal{R} -linear mapping from \mathcal{A} into $\mathcal{Z}(\mathcal{A})$. It was Brešar who initiated the study of commuting traces and centralizing traces of bilinear mappings in his series of works [10, 11, 12, 13, 15], where he investigated the structure of commuting traces and centralizing traces of (bi-)linear mappings on prime rings. It has turned out that in certain rings, in particular, prime rings of characteristic different from 2 and 3, every centralizing trace of a biadditive mapping is commuting. Moreover, every centralizing mapping of a prime ring of characteristic not 2 is of the proper form and is actually commuting. Lee et al further generalized Brešar's results by showing that each commuting trace of an arbitrary multilinear mapping on a prime ring also has the proper form [28].

Cheung in [21] studied commuting mappings of triangular algebras (e.g., of upper triangular matrix algebras and nest algebras). He determined the class of triangular algebras for which every commuting mapping is proper. Xiao and Wei [49] extended Cheung's result to the generalized matrix algebra case. Motivated by the results of Brešar and Cheung, Benkovič and Eremita [8] considered commuting traces of bilinear mappings on a triangular algebra $\begin{bmatrix} A & M \\ O & B \end{bmatrix}$. They gave conditions under which every commuting trace of a triangular algebra $\begin{bmatrix} A & M \\ O & B \end{bmatrix}$ is proper. In view of the above works, it is natural and necessary to characterize centralizing traces of (multi-)linear mappings on triangular algebras. One of the main aims of this article is to provide a sufficient condition for each centralizing trace of an arbitrary bilinear mapping on a triangular algebra $\begin{bmatrix} A & M \\ O & B \end{bmatrix}$ to be proper.

Another important purpose of this article is to address the Lie triple isomorphisms problem of triangular algebras. At his 1961 AMS Hour Talk, Herstein proposed many problems concerning the structure of Jordan and Lie mappings in associative simple and prime rings [26]. The renowned Herstein's Lie-type mappings research program was formulated since then. The involved Lie mappings mainly include Lie isomorphisms, Lie triple isomorphisms, Lie derivations and Lie triple derivations et al. Given a commutative ring \mathcal{R} with identity and two associative \mathcal{R} -algebras \mathcal{A} and \mathcal{B} , one define a *Lie triple isomorphism* from \mathcal{A} into \mathcal{B} to be an \mathcal{R} -linear bijective mapping \mathfrak{l} satisfying the condition

$$\mathfrak{l}([a, b], c) = [\mathfrak{l}(a), \mathfrak{l}(b)], \mathfrak{l}(c) \quad \forall a, b, c \in \mathcal{A}.$$

For example, an isomorphism or a negative of an anti-isomorphism of one algebra onto another is also a Lie isomorphism. Furthermore, every Lie isomorphism and every Jordan isomorphism are Lie triple isomorphisms. One can ask whether the converse is true in some special cases. That is, does every Lie triple isomorphism between certain associative algebras arise from isomorphisms and anti-isomorphisms

in the sense of modulo mappings whose range is central ? Recall that a Lie isomorphism $\mathfrak{l}: A \longrightarrow B$ is *standard* if

$$\mathfrak{l} = \mathfrak{m} + \mathfrak{n}, \quad (\clubsuit)$$

where \mathfrak{m} is an isomorphism or the negative of an anti-isomorphism from \mathcal{A} onto \mathcal{B} and $\mathfrak{n}: \mathcal{A} \longrightarrow \mathcal{Z}(\mathcal{B})$ is an \mathcal{R} -linear mapping annihilating all commutators. We say that a Lie triple isomorphism $\mathfrak{l}: A \longrightarrow B$ is *standard* if

$$\mathfrak{l} = \pm \mathfrak{m} + \mathfrak{n}, \quad (\spadesuit)$$

where \mathfrak{m} is an isomorphism or an anti-isomorphism from \mathcal{A} onto \mathcal{B} and $\mathfrak{n}: \mathcal{A} \longrightarrow \mathcal{Z}(\mathcal{B})$ is an \mathcal{R} -linear mapping annihilating all second commutators.

The resolution of Herstein's Lie isomorphisms problem in matrix algebra background has been well-known for a long time. Hua [27] proved that every Lie automorphism of the full matrix algebra $\mathcal{M}_n(\mathcal{D})$ ($n \geq 3$) over a division ring \mathcal{D} is of the standard form (\clubsuit) . This result was extended to the nonlinear case by Dolinar [24] and was further refined by Šemrl [44]. Doković [23] showed that every Lie automorphism of upper triangular matrix algebras $\mathcal{T}_n(\mathcal{R})$ over a commutative ring \mathcal{R} without nontrivial idempotents has the standard form as well. Marcoux and Sourour [33] classified the linear mappings preserving commutativity in both directions (i.e., $[x, y] = 0$ if and only if $[\mathfrak{f}(x), \mathfrak{f}(y)] = 0$) on upper triangular matrix algebras $\mathcal{T}_n(\mathbb{F})$ over a field \mathbb{F} . Such a mapping is either the sum of an algebra automorphism of $\mathcal{T}_n(\mathbb{F})$ (which is inner) and a mapping into the center $\mathbb{F}I$, or the sum of the negative of an algebra anti-automorphism and a mapping into the center $\mathbb{F}I$. The classification of the Lie automorphisms of $\mathcal{T}_n(\mathbb{F})$ is obtained as a consequence. Benkovič and Eremita [8] applied the theory of commuting traces to study the Lie isomorphisms on a triangular algebra. They provided sufficient conditions under which every commuting trace of triangular algebra $\begin{bmatrix} A & M \\ O & B \end{bmatrix}$ is proper. It also turns out that under some mild assumptions, each Lie isomorphism of $\begin{bmatrix} A & M \\ O & B \end{bmatrix}$ has the standard form (\clubsuit) . Calderón Martín and Martín González observed that every Lie triple isomorphism of the full matrix algebra $\mathcal{M}_n(\mathbb{C})$ over the complex field \mathbb{C} is of the standard form (\spadesuit) [18]. Simultaneously, Lie triple isomorphisms between rings and between (non-)self-adjoint operator algebras have received a fair amount of attentions. The involved rings and operator algebras include (semi-)prime rings, the algebra of bounded linear operators, C^* -algebras, von Neumann algebras, H^* -algebras, nest algebras, reflexive algebras and so on, see [16, 17, 18, 19, 31, 32, 34, 35, 37, 38, 39, 42, 43, 44, 45, 46, 51, 52].

This is the second paper in a series of three that we are planning on this topic. The first paper was dedicated to studying, in more details, commuting traces and Lie isomorphisms on generalized matrix algebras [50]. This article is organized as following. Section 2 contains the definition of triangular algebra and some classical examples. In Section 3 we provide sufficient conditions for each centralizing trace of arbitrary bilinear mappings on a triangular algebra $\begin{bmatrix} A & M \\ O & B \end{bmatrix}$ to be proper (Theorem 3.4). And then we apply this result to describe the centralizing traces of bilinear mappings on certain classical triangular algebras. In Section 4 we will give sufficient conditions under which every Lie triple isomorphism from a triangular algebra into another one has the almost standard form (Theorem 4.4). As corollaries of Theorem 4.4, characterizations of Lie triple isomorphisms on several kinds of triangular algebras are obtained. The last section contains some potential future research topics related to our current work.

2. PRELIMINARIES

Let \mathcal{R} be a commutative ring with identity. Let A and B be unital algebras over \mathcal{R} . Recall that an (A, B) -bimodule M is *loyal* if $aMb = 0$ implies that $a = 0$ or $b = 0$ for any $a \in A, b \in B$. Clearly, each loyal (A, B) -bimodule M is faithful as a left A -module and also as a right B -module.

Let A, B be unital associative algebras over \mathcal{R} and M be a unital (A, B) -bimodule, which is faithful as a left A -module and also as a right B -module. We denote the *triangular algebra* consisting of A, B and M by

$$\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}.$$

Then \mathcal{T} is an associative and noncommutative \mathcal{R} -algebra. The center $\mathcal{Z}(\mathcal{T})$ of \mathcal{T} is (see [21, Proposition 3])

$$\mathcal{Z}(\mathcal{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| am = mb, \forall m \in M \right\}.$$

Let us define two natural \mathcal{R} -linear projections $\pi_A : \mathcal{T} \rightarrow A$ and $\pi_B : \mathcal{T} \rightarrow B$ by

$$\pi_A : \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \mapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \mapsto b.$$

It is easy to see that $\pi_A(\mathcal{Z}(\mathcal{T}))$ is a subalgebra of $\mathcal{Z}(A)$ and that $\pi_B(\mathcal{Z}(\mathcal{T}))$ is a subalgebra of $\mathcal{Z}(B)$. Furthermore, there exists a unique algebraic isomorphism $\tau : \pi_A(\mathcal{Z}(\mathcal{T})) \rightarrow \pi_B(\mathcal{Z}(\mathcal{T}))$ such that $am = m\tau(a)$ for all $a \in \pi_A(\mathcal{Z}(\mathcal{T}))$ and for all $m \in M$.

Let 1 (resp. $1'$) be the identity of the algebra A (resp. B), and let I be the identity of the triangular algebra \mathcal{T} . We will use the following notations:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = I - P = \begin{bmatrix} 0 & 0 \\ 0 & 1' \end{bmatrix}$$

and

$$\mathcal{T}_{11} = P\mathcal{T}P, \quad \mathcal{T}_{12} = P\mathcal{T}Q, \quad \mathcal{T}_{22} = Q\mathcal{T}Q.$$

Thus the triangular algebra \mathcal{T} can be written as

$$\mathcal{T} = P\mathcal{T}P + P\mathcal{T}Q + Q\mathcal{T}Q = \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{22}.$$

\mathcal{T}_{11} and \mathcal{T}_{22} are subalgebras of \mathcal{T} which are isomorphic to A and B , respectively. \mathcal{T}_{12} is a $(\mathcal{T}_{11}, \mathcal{T}_{22})$ -bimodule which is isomorphic to the (A, B) -bimodule M . It should be remarked that $\pi_A(\mathcal{Z}(\mathcal{T}))$ and $\pi_B(\mathcal{Z}(\mathcal{T}))$ are isomorphic to $P\mathcal{Z}(\mathcal{T})P$ and $Q\mathcal{Z}(\mathcal{T})Q$, respectively. Then there is an algebra isomorphism $\tau : P\mathcal{Z}(\mathcal{T})P \rightarrow Q\mathcal{Z}(\mathcal{T})Q$ such that $am = m\tau(a)$ for all $m \in P\mathcal{T}Q$.

Let us list some classical examples of triangular algebras and matrix algebras which will be revisited in the sequel (Section 3, Section 4 and Section 5). Since these examples have already been presented in many papers, we just state their titles without any introduction. We refer the reader to [8, 29, 49] for more details.

- (a) Upper and lower triangular matrix algebras;
- (b) Block upper and lower triangular matrix algebras;
- (c) Hilbert space nest algebras;
- (d) Full matrix algebras;
- (e) Inflated algebras.

3. CENTRALIZING TRACES OF TRIANGULAR ALGEBRAS

In this section we will establish sufficient conditions for each commuting trace of arbitrary bilinear mappings on a triangular algebra $\begin{bmatrix} A & M \\ O & B \end{bmatrix}$ to be proper (Theorem 3.4). Consequently, we are able to describe centralizing traces of bilinear mappings on upper triangular matrix algebras and nest algebras. The most important fact is that Theorem 3.4 will be used to characterize Lie triple isomorphisms from a triangular algebra into another in Section 4.

We now list some basic facts related to triangular algebras, which can be found in [8, Section 2].

Lemma 3.1. *Let M be a loyal (A, B) -bimodule and let $f, g: M \rightarrow A$ be arbitrary mappings. Suppose $f(m)n + g(n)m = 0$ for all $m, n \in M$. If B is noncommutative, then $f = g = 0$.*

Lemma 3.2. *Let $\mathcal{T} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ be a triangular algebra with a loyal (A, B) -bimodule M , $\lambda \in \pi_B(\mathcal{Z}(\mathcal{T}))$ and $b \in B$ be a nonzero element. If $\lambda b = 0$, then $\lambda = 0$.*

Lemma 3.3. *Let $\mathcal{T} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ be a triangular algebra with a loyal (A, B) -bimodule M . Then the center $\mathcal{Z}(\mathcal{T})$ of \mathcal{T} is a domain.*

We are in position to state the main theorem of this section.

Theorem 3.4. *Let $\mathcal{T} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ be a 2-torsion free triangular algebra over the commutative ring \mathcal{R} and $\mathfrak{q}: \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ be an \mathcal{R} -bilinear mapping. If*

- (1) *each commuting linear mapping on A or B is proper,*
- (2) *$\pi_A(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(A) \neq A$ and $\pi_B(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(B) \neq B$,*
- (3) *M is loyal,*

then every centralizing trace $\mathfrak{T}_{\mathfrak{q}}: \mathcal{T} \rightarrow \mathcal{T}$ of \mathfrak{q} is proper.

For convenience, let us write $A_1 = A$, $A_2 = B$ and $A_3 = M$. We denote the unity of A_1 by 1 and the unity of A_2 by $1'$. Suppose that $\mathfrak{T}_{\mathfrak{q}}$ is an arbitrary trace of the \mathcal{R} -bilinear mapping \mathfrak{q} . Then there exist bilinear mappings $f_{ij}: A_i \times A_j \rightarrow A_1$, $g_{ij}: A_i \times A_j \rightarrow A_2$ and $h_{ij}: A_i \times A_j \rightarrow A_3$ ($1 \leq i \leq j \leq 3$) such that

$$\mathfrak{T}_{\mathfrak{q}}: \begin{bmatrix} a_1 & a_3 \\ & a_2 \end{bmatrix} \mapsto \begin{bmatrix} F(a_1, a_2, a_3) & H(a_1, a_2, a_3) \\ & G(a_1, a_2, a_3) \end{bmatrix},$$

where

$$\begin{aligned} F(a_1, a_2, a_3) &= \sum_{1 \leq i \leq j \leq 3} f_{ij}(a_i, a_j), \\ G(a_1, a_2, a_3) &= \sum_{1 \leq i \leq j \leq 3} g_{ij}(a_i, a_j), \\ H(a_1, a_2, a_3) &= \sum_{1 \leq i \leq j \leq 3} h_{ij}(a_i, a_j). \end{aligned}$$

Since $\mathfrak{T}_{\mathfrak{q}}$ is centralizing, we have

$$\left[\begin{bmatrix} F & H \\ & G \end{bmatrix}, \begin{bmatrix} a_1 & a_3 \\ & a_2 \end{bmatrix} \right] = \begin{bmatrix} [F, a_1] & Fa_3 + Ha_2 - a_1H - a_3G \\ & [G, a_2] \end{bmatrix} \in \mathcal{Z}(\mathcal{T}). \quad (3.1)$$

Now we divide the proof of Theorem 3.4 into a series of lemmas for comfortable reading.

Lemma 3.5. *Let $K : A_2 \times A_2 \rightarrow A_3$ (resp. $K : A_1 \times A_1 \rightarrow A_3$) be an \mathcal{R} -bilinear mapping. If $K(x, x)x = 0$ (resp. $xK(x, x) = 0$) for all $x \in A_2$ (resp. for all $x \in A_1$), then $K(x, x) = 0$.*

Proof. Setting $x = 1'$, we obtain that $K(1', 1') = 0$. Replacing x by $x + 1'$ in $K(x, x)x = 0$, we get

$$K(x, x) = -(K(1', x) + K(x, 1'))(1' + x). \quad (3.2)$$

Substituting $x - 1'$ for x in $K(x, x)x = 0$, we arrive at

$$K(x, x) = (K(1', x) + K(x, 1'))(1' - x). \quad (3.3)$$

Combining the above two relations gives $K(1', x) + K(x, 1') = 0$. Thus $K(x, x) = 0$. \square

Lemma 3.6. $H(a_1, a_2, a_3) = h_{13}(a_1, a_3) + h_{23}(a_2, a_3) + h_{33}(a_3, a_3)$.

Proof. It follows from the matrix relation (3.1) that

$$Fa_3 + Ha_2 - a_1H - a_3G = 0. \quad (3.4)$$

Let us take $a_1 = 0$ and $a_2 = 0$ into (3.4). Then (3.1) implies that

$$f_{33}(a_3, a_3)a_3 = a_3g_{33}(a_3, a_3) \quad (3.5)$$

for all $a_3 \in A_3$. Let us choose $a_1 = 0$ and $a_3 = 0$ in (3.4). Then $0 = Ha_2 = h_{22}(a_2, a_2)a_2$ for all $a_2 \in A_2$. In view of Lemma 3.5, we have $h_{22}(a_2, a_2) = 0$. Similarly, putting $a_2 = 0$ and $a_3 = 0$ in (3.4) yields $h_{11}(a_1, a_1) = 0$ for all $a_1 \in A_1$. Furthermore, setting $a_3 = 0$ in (3.4), we see that

$$(h_{12}(a_1, a_2)a_2 - a_1h_{12}(a_1, a_2)) = 0$$

for all $a_1 \in A_1, a_2 \in A_2$. Replacing a_1 by $-a_1$ in the above relation and comparing the obtained two relations gives $a_1h_{12}(a_1, a_2) = 0$ for all $a_1 \in A_1, a_2 \in A_2$. In particular, $h_{12}(1, a_2) = 0$ for all $a_2 \in A_2$. Substituting $a_1 + 1$ for a_1 in $a_1h_{12}(a_1, a_2) = 0$ leads to $h_{12}(a_1, a_2) = 0$ for all $a_1 \in A_1, a_2 \in A_2$. Therefore

$$H(a_1, a_2, a_3) = h_{13}(a_1, a_3) + h_{23}(a_2, a_3) + h_{33}(a_3, a_3)$$

for all $a_1 \in A_1, a_2 \in A_2, a_3 \in A_3$. \square

Lemma 3.7. *With notations as above, we have*

- (1) $a_1 \mapsto f_{11}(a_1, a_1)$ is a commuting trace,
 $a_1 \mapsto f_{13}(a_1, a_3)$ is a commuting linear mapping for each $a_3 \in A_3$,
 $a_2 \mapsto g_{22}(a_2, a_2)$ is a commuting trace,
 $a_2 \mapsto g_{23}(a_2, a_3)$ is a commuting linear mapping for each $a_3 \in A_3$,
- (2) $[g_{11}(a_1, a_1), a_2] = \tau([f_{12}(a_1, a_2), a_1]) \in \mathcal{Z}(A_2)$,
 $[g_{12}(a_1, a_2), a_2] = \tau([f_{22}(a_2, a_2), a_1]) \in \mathcal{Z}(A_2)$,
 $[g_{13}(a_1, a_3), a_2] = \tau([f_{23}(a_2, a_3), a_1]) \in \mathcal{Z}(A_2)$,
- (3) $f_{33}(a_3, a_3) \in \mathcal{Z}(A_1)$ and $g_{33}(a_3, a_3) \in \mathcal{Z}(A_2)$.

Proof. By the relation (3.1) we know that

$$\tau([F, a_1]) = [G, a_2]. \quad (3.6)$$

Let us take $a_1 = 0$ in (3.6). Then

$$[g_{22}(a_2, a_2) + g_{23}(a_2, a_3) + g_{33}(a_3, a_3), a_2] = 0 \quad (3.7)$$

for all $a_2 \in A_2, a_3 \in A_3$. Replacing a_3 by $-a_3$ in (3.7) we get

$$[g_{22}(a_2, a_2) + g_{33}(a_3, a_3), a_2] = 0 \quad (3.8)$$

for all $a_2 \in A_2, a_3 \in A_3$. Putting $a_3 = 0$ in (3.7) and combining (3.7) and (3.8), we obtain

$$[g_{22}(a_2, a_2), a_2] = 0, \quad [g_{23}(a_2, a_3), a_2] = 0, \quad [g_{33}(a_3, a_3), a_2] = 0$$

for all $a_2 \in A_2, a_3 \in A_3$. In a similar way, we have

$$[f_{11}(a_1, a_1), a_1] = 0, \quad [f_{33}(a_3, a_3), a_1] = 0, \quad [f_{13}(a_1, a_3), a_1] = 0.$$

Setting $a_3 = 0$ in (3.6), we arrive at

$$\tau([f_{12}(a_1, a_2) + f_{22}(a_2, a_2), a_1]) = [g_{11}(a_1, a_1) + g_{12}(a_1, a_2), a_2] \quad (3.9)$$

for all $a_1 \in A_1, a_2 \in A_2$. Replacing a_1 by $-a_1$ in (3.9) and then comparing the obtained relation with (3.9), we get

$$\tau([f_{22}(a_2, a_2), a_1]) = [g_{12}(a_1, a_2), a_2] \in \mathcal{Z}(A_2) \quad (3.10)$$

and

$$\tau([f_{12}(a_1, a_2), a_1]) = [g_{11}(a_1, a_1), a_2] \in \mathcal{Z}(A_2) \quad (3.11)$$

for all $a_1 \in A_1, a_2 \in A_2$. In view of (3.6), (3.10), (3.11) we conclude

$$\tau([f_{23}(a_2, a_3), a_1]) = [g_{13}(a_1, a_3), a_2] \in \mathcal{Z}(A_2)$$

for all $a_1 \in A_1, a_2 \in A_2, a_3 \in A_3$. \square

Lemma 3.8. *There exist a linear mapping $\xi : A_3 \rightarrow \mathcal{Z}(A_2)$ and a bilinear mapping $\eta : A_2 \times A_3 \rightarrow \mathcal{Z}(A_2)$ such that $g_{23}(a_2, a_3) = \xi(a_3)a_2 + \eta(a_2, a_3)$.*

Proof. Since $a_2 \mapsto g_{23}(a_2, a_3)$ is a commuting linear mapping for each $a_3 \in A_3$, then by the hypothesis (1) there exist mappings $\xi : A_3 \rightarrow \mathcal{Z}(A_2)$ and $\eta : A_2 \times A_3 \rightarrow \mathcal{Z}(A_2)$ such that

$$g_{23}(a_2, a_3) = \xi(a_3)a_2 + \eta(a_2, a_3),$$

where η is \mathcal{R} -linear in the first argument. Let us show that ξ is \mathcal{R} -linear and η is \mathcal{R} -bilinear. Clearly,

$$\begin{aligned} g_{23}(a_2, a_3 + b_3) &= \xi(a_3 + b_3)a_2 + \eta(a_2, a_3 + b_3) \\ g_{23}(a_2, a_3) + g_{23}(a_2, b_3) &= \xi(a_3)a_2 + \eta(a_2, a_3) + \xi(b_3)a_2 + \eta(a_2, b_3) \end{aligned}$$

for all $a_2 \in A_2, a_3, b_3 \in A_3$. So

$$(\xi(a_3 + b_3) - \xi(a_3) - \xi(b_3))a_2 + \eta(a_2, a_3 + b_3) - \eta(a_2, a_3) - \eta(a_2, b_3) = 0$$

for all $a_2 \in A_2, a_3, b_3 \in A_3$. Note that ξ and η map into $\mathcal{Z}(A_2)$. Hence $(\xi(a_3 + b_3) - \xi(a_3) - \xi(b_3))[a_2, b_2] = 0$ for all $a_2, b_2 \in A_2$, and $a_3, b_3 \in A_3$. Note that A_2 is noncommutative. Applying Lemma 3.2 yields that ξ is \mathcal{R} -linear mapping. Consequently, η is \mathcal{R} -linear in the second argument. \square

Lemma 3.9. *$f_{23}(a_2, a_3) \in \mathcal{Z}(A_1)$ and $g_{13}(a_1, a_3) \in \mathcal{Z}(A_2)$.*

Proof. By Lemma 3.7 it is enough to prove $f_{23}(a_2, a_3) \in \mathcal{Z}(A_1)$. Setting $a_1 = 0$ in (3.4) and using (3.5), we obtain

$$\begin{aligned} (f_{22}(a_2, a_2) + f_{23}(a_2, a_3))a_3 + (h_{33}(a_3, a_3) + h_{23}(a_2, a_3))a_2 \\ - a_3(g_{22}(a_2, a_2) + g_{23}(a_2, a_3)) = 0 \end{aligned} \quad (3.12)$$

for all $a_2 \in A_2, a_3 \in A_3$. Replacing a_2 by $-a_2$ in the equation (3.12) and then comparing with it, we get

$$h_{23}(a_2, a_3)a_2 = a_3g_{22}(a_2, a_2) - f_{22}(a_2, a_2)a_3 \quad (3.13)$$

and

$$h_{33}(a_3, a_3)a_2 = a_3g_{23}(a_2, a_3) - f_{23}(a_2, a_3)a_3 \quad (3.14)$$

for all $a_2 \in A_2, a_3 \in A_3$. Note that $[g_{23}(a_2, a_3), a_2] = 0$ for all $a_2 \in A_2, a_3 \in A_3$. Replacing a_2 by $a_2 + 1'$ in $[g_{23}(a_2, a_3), a_2] = 0$ gives $g_{23}(1', a_3) \in \mathcal{Z}(A_2)$. On the other hand, Lemma 3.7 shows that $f_{23}(1', a_3) \in \mathcal{Z}(A_1)$ for all $a_3 \in A_3$. Taking $a_2 = 1'$ in (3.14) we have

$$h_{33}(a_3, a_3) = a_3\alpha(a_3), \quad (3.15)$$

where $\alpha(a_3) = g_{23}(1', a_3) - \tau(f_{23}(1', a_3)) \in \mathcal{Z}(A_2)$. It follows from (3.14), (3.15) and Lemma 3.8 that

$$a_3(\alpha(a_3) - \xi(a_3))a_2 = (\tau^{-1}(\eta(a_2, a_3)) - f_{23}(a_2, a_3))a_3. \quad (3.16)$$

We denote $Y(a_3) = \alpha(a_3) - \xi(a_3)$, $X(a_2, a_3) = \tau^{-1}(\eta(a_2, a_3)) - f_{23}(a_2, a_3)$. Taking $a_2 = 1'$ into (3.16), we see that $(\tau^{-1}(Y(a_3)) - X(1', a_3))a_3 = 0$ for all $a_3 \in A_3$.

We claim that

$$Y(a_3) = \tau(X(1', a_3)) \quad (3.17)$$

for all $a_3 \in A_3$. In fact, replacing a_3 by $m + n$ in $(\tau^{-1}(Y(a_3)) - X(1', a_3))a_3 = 0$, we get

$$(\tau^{-1}(Y(m)) - X(1', m))n + (\tau^{-1}(Y(n)) - X(1', n))m = 0$$

for all $m, n \in A_3$. Applying Lemma 3.1 yields $Y(m) = \tau(X(1', m))$ for all $m \in A_3$. Thus our claim follows.

Now let us rewrite the relation (3.16) as

$$a_3\tau(X(1', a_3))a_2 = X(a_2, a_3)a_3 \quad (3.18)$$

for all $a_3 \in A_3$. Replacing a_3 by $m + n$ in (3.18), we obtain

$$m\tau(X(1', n))a_2 + n\tau(X(1', m))a_2 = X(a_2, n)m + X(a_2, m)n \quad (3.19)$$

for all $a_2 \in A_2, m, n \in A_3$. Replacing n by a_1n in (19) and then subtracting the left multiplication of (3.19) by a_1 , we arrive at

$$\begin{aligned} m\tau(X(1', a_1n))a_2 - a_1m\tau(X(1', n))a_2 \\ = X(a_2, m)a_1n + X(a_2, a_1n)m - a_1X(a_2, m)n - a_1X(a_2, n)m \end{aligned} \quad (3.20)$$

for all $a_1 \in A_1, a_2 \in A_2$ and $m, n \in A_3$. Taking $m = n$ in (3.20) and using (3.18), we have

$$m\tau(X(1', a_1m))a_2 = X(a_2, a_1m)m + [X(a_2, m), a_1]m \quad (3.21)$$

for all $a_1 \in A_1, a_2 \in A_2, m \in A_3$. Left multiplying a_1 in (3.21) and considering (3.18), we get $[X(a_2, a_1m), a_1]m = a_1[X(a_2, m), a_1]m$. That is,

$$([X(a_2, a_1m), a_1] - a_1[X(a_2, m), a_1])m = 0$$

for all $a_1 \in A_1, a_2 \in A_2$ and $m \in A_3$. Let us write $P(m) = [X(a_2, a_1m), a_1] - a_1[X(a_2, m), a_1]$ for some fixed a_1, a_2 . Then $P: A_3 \rightarrow A_1$ is an \mathcal{R} -linear mapping for each $a_1 \in A_1, a_2 \in A_2$, and $P(m)m = 0$. A linearization of $P(m)m = 0$ shows $P(m)n + P(n)m = 0$ for all $m, n \in A_3$. In view of Lemma 3.1 we know that $P(m) = 0$. So

$$[X(a_2, a_1m), a_1] = a_1[X(a_2, m), a_1]$$

for all $a_1 \in A_1, a_2 \in A_2, m \in A_3$. Picking $b_1 \in A_1$ such that $[a_1, b_1] \neq 0$, and then commuting with b_1 , we get $[a_1, b_1][X(a_2, m), a_1] = 0$ since $[X(a_2, m), a_1] = [a_1, f_{23}(a_2, m)] \in \mathcal{Z}(A_1)$ by Lemma 3.7. Thus Lemma 3.2 implies $[X(a_2, m), a_1] = [a_1, f_{23}(a_2, a_3)] = 0$ and this completes the proof of the lemma. \square

Lemma 3.10. *With notations as above, we have*

- (1) $f_{22}(a_2, a_2) \in \mathcal{Z}(A_1)$ and $g_{11}(a_1, a_1) \in \mathcal{Z}(A_2)$;
- (2) $a_1 \mapsto f_{12}(a_1, a_2)$ is a commuting linear mapping for each $a_2 \in A_2$,
 $a_2 \mapsto g_{12}(a_1, a_2)$ is a commuting linear mapping for each $a_1 \in A_1$.

Proof. Taking $a_2 = 0$ in (3.4) and using (3.5), we get

$$\begin{aligned} & (f_{11}(a_1, a_1) + f_{13}(a_1, a_3))a_3 - a_3(g_{11}(a_1, a_1) + g_{13}(a_1, a_3)) \\ & - a_1(h_{13}(a_1, a_3) + h_{33}(a_3, a_3)) = 0 \end{aligned} \quad (3.22)$$

for all $a_1 \in A_1, a_3 \in A_3$. Note that \mathcal{R} is 2-torsion free ring. Substituting $-a_1$ for a_1 in (3.22), we obtain

$$a_1 h_{13}(a_1, a_3) = f_{11}(a_1, a_1)a_3 - a_3 g_{11}(a_1, a_1) \quad (3.23)$$

for all $a_1 \in A_1, a_3 \in A_3$. Combining (3.22) with (3.23) gives

$$a_1 h_{33}(a_3, a_3) = f_{13}(a_1, a_3)a_3 - a_3 g_{13}(a_1, a_3) \quad (3.24)$$

for all $a_1 \in A_1, a_3 \in A_3$. On the other hand, replacing a_3 by $a_1 a_3$ in (3.13) and subtracting the left multiplication of (3.13) by a_1 we get

$$(a_1 h_{23}(a_2, a_3) - h_{23}(a_2, a_1 a_3))a_2 = [f_{22}(a_2, a_2), a_1]a_3 \quad (3.25)$$

for all $a_1 \in A_1, a_2 \in A_2, a_3 \in A_3$. Replacing a_3 by $a_3 a_2$ in (3.13) and subtracting the right multiplication of (3.11) by a_2 we get $h_{23}(a_2, a_3 a_2)a_2 = h_{23}(a_2, a_3)a_2 a_2$. Let us set $K(x, y) = h_{23}(x, a_3 y) - h_{23}(x, a_3)y$, where $x, y \in A_2$. It is easy to see that $K(x, y): A_2 \times A_2 \rightarrow A_3$ is an \mathcal{R} -bilinear mapping, and $K(a_2, a_2)a_2 = 0$. It follows from Lemma 3.5 that

$$h_{23}(a_2, a_3 a_2) = h_{23}(a_2, a_3)a_2 \quad (3.26)$$

for all $a_2 \in A_2, a_3 \in A_3$. Substituting $a_3 a_2$ for a_3 in (3.23) and then subtracting the right multiplication of (3.23) by a_2 , we have

$$a_3[g_{11}(a_1, a_1), a_2] = a_1(h_{13}(a_1, a_3 a_2) - h_{13}(a_1, a_3)a_2) \quad (3.27)$$

for all $a_1 \in A_1, a_2 \in A_2$ and $a_3 \in A_3$. Combining the relations (3.13) – (3.14), (3.23) – (3.24) together with (3.4) yields

$$a_1 h_{23}(a_2, a_3) + a_3 g_{12}(a_1, a_2) = h_{13}(a_1, a_3)a_2 + f_{12}(a_1, a_2)a_3 \quad (3.28)$$

for all $a_1 \in A_1, a_2 \in A_2$ and $a_3 \in A_3$. Replacing a_3 by $a_3 a_2$ in (3.28) and then subtracting the right multiplication of (3.28) by a_2 , we arrive at

$$\begin{aligned} & a_1(h_{23}(a_2, a_3)a_2 - h_{23}(a_2, a_3 a_2)) + a_3[g_{12}(a_1, a_2), a_2] \\ & = (h_{13}(a_1, a_3)a_2 - h_{13}(a_1, a_3 a_2))a_2 \end{aligned} \quad (3.29)$$

for all $a_1 \in A_1, a_2 \in A_2$ and $a_3 \in A_3$. Considering the identities (3.26) and (3.29), we get

$$-a_3[g_{12}(a_1, a_2), a_2] = (h_{13}(a_1, a_3 a_2) - h_{13}(a_1, a_3)a_2)a_2 \quad (3.30)$$

for all $a_1 \in A_1, a_2 \in A_2$ and $a_3 \in A_3$. Making the right multiplication of (3.27) by a_2 and then subtracting the left multiplication of (3.30) by a_1 , we obtain

$$a_1 a_3 [g_{12}(a_1, a_2), a_2] = a_3 [a_2, g_{11}(a_1, a_1)]a_2$$

for all $a_1 \in A_1, a_2 \in A_2$ and $a_3 \in A_3$. According to (3.10), we have

$$a_1[f_{22}(a_2, a_2), a_1]a_3 = a_3[a_2, g_{11}(a_1, a_1)]a_2$$

for all $a_1 \in A_1, a_2 \in A_2$ and $a_3 \in A_3$. Therefore

$$\begin{bmatrix} a_1[f_{22}(a_2, a_2), a_1] & 0 \\ 0 & [a_2, g_{11}(a_1, a_1)]a_2 \end{bmatrix} \in \mathcal{Z}(\mathcal{T}).$$

Commuting with $b_2 \in A_2$, we get $[g_{11}(a_1, a_1), a_2][a_2, b_2] = 0$. Then Lemma 3.2 implies $g_{11}(a_1, a_1) \in \mathcal{Z}(A_2)$ and hence $a_1 \mapsto f_{12}(a_1, a_2)$ is a commuting linear mapping for each $a_2 \in A_2$ by Lemma 3.7. Similarly, we have $f_{22}(a_2, a_2) \in \mathcal{Z}(A_1)$ and $a_2 \mapsto g_{12}(a_1, a_2)$ is a commuting linear mapping for each $a_1 \in A_1$. \square

Proof of Theorem 3.4. Let $\mathbf{q}: \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ be an arbitrary \mathcal{R} -bilinear mapping of \mathcal{T} . It follows from Lemma 3.7, Lemma 3.9 and Lemma 3.10 that every centralizing trace of \mathbf{q} is commuting. Then the desired result can be obtained by [8, Theorem 3.1]. \square

An algebra \mathcal{A} over a commutative ring \mathcal{R} is said to be *central* over \mathcal{R} if $\mathcal{Z}(\mathcal{A}) = \mathcal{R}1$. The following technical lemma will be used to deal with the centralizing traces of upper triangular matrix algebras.

Lemma 3.11. *Let $\mathcal{T} = \begin{bmatrix} \mathcal{R} & M \\ 0 & B \end{bmatrix}$ be a 2-torsion free triangular algebra over the commutative ring \mathcal{R} and $\mathbf{q}: \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}$ be an \mathcal{R} -bilinear mapping. Suppose that B is noncommutative and both \mathcal{T} and B are central over \mathcal{R} . If*

- (1) *each commuting linear mapping on B is proper,*
- (2) *for any $r \in \mathcal{R}$ and $m \in M$, $rm = 0$ implies $r = 0$ or $m = 0$,*
- (3) *there exist $m_0 \in M$ and $b_0 \in B$ such that $m_0 b_0$ and m_0 are linearly independent over \mathcal{R} ,*

then each centralizing trace $\mathfrak{T}_{\mathbf{q}}: \mathcal{T} \longrightarrow \mathcal{T}$ of \mathbf{q} is proper.

Proof. We use the same notations of Theorem 3.4. Since $A_1 = \mathcal{R}$ is commutative, then the equation (3.1) shows that $[F, a_1] = 0$ and hence $[G, a_2] = 0$. Therefore the centralizing trace $\mathfrak{T}_{\mathbf{q}}$ is commuting. Now the desired result follows from [8, Lemma 3.2]. \square

Corollary 3.12. *Let \mathcal{R} be a 2-torsion free commutative domain and $\mathcal{T}_n(\mathcal{R}) (n \geq 2)$ be the algebra of all $n \times n$ upper triangular matrices over \mathcal{R} . Suppose that $\mathbf{q}: \mathcal{T}_n(\mathcal{R}) \times \mathcal{T}_n(\mathcal{R}) \longrightarrow \mathcal{T}_n(\mathcal{R})$ is an \mathcal{R} -bilinear mapping. Then every centralizing trace $\mathfrak{T}_{\mathbf{q}}: \mathcal{T}_n(\mathcal{R}) \longrightarrow \mathcal{T}_n(\mathcal{R})$ of \mathbf{q} is proper.*

Proof. The proof is similar with that of [8, Corollary 3.4] and hence we omit it here. \square

Applying Theorem 3.4 and [8, Corollary 3.5] yields

Corollary 3.13. *Let H be a Hilbert space, \mathcal{N} be a nest of H and $\mathcal{Alg}(\mathcal{N})$ be the nest algebra associated with \mathcal{N} . Suppose that $\mathbf{q}: \mathcal{Alg}(\mathcal{N}) \times \mathcal{Alg}(\mathcal{N}) \longrightarrow \mathcal{Alg}(\mathcal{N})$ is an \mathcal{R} -bilinear mapping. Then every centralizing trace $\mathfrak{T}_{\mathbf{q}}: \mathcal{Alg}(\mathcal{N}) \longrightarrow \mathcal{Alg}(\mathcal{N})$ of \mathbf{q} is proper.*

4. LIE TRIPLE ISOMORPHISMS ON TRIANGULAR ALGEBRAS

Lemma 4.1. *Let \mathcal{R} be 2-torsion free. Then the triangular algebra $\mathcal{T} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ does not contain nonzero central Jordan ideals.*

Proof. Let \mathcal{J} be a central Jordan ideal of \mathcal{T} . Suppose that $\begin{bmatrix} \alpha & 0 \\ 0 & \tau(\alpha) \end{bmatrix} \in \mathcal{J}$. Hence

$$\begin{bmatrix} \alpha & 0 \\ 0 & \tau(\alpha) \end{bmatrix} \circ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha m + m\tau(\alpha) \\ 0 & 0 \end{bmatrix}$$

for all $m \in M$. This implies that $2\alpha M = 0$ and so $\alpha = 0 = \begin{bmatrix} \alpha & 0 \\ 0 & \tau(\alpha) \end{bmatrix}$. \square

Theorem 4.2. *Let $\mathcal{T} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ and $\mathcal{T}' = \begin{bmatrix} A' & M' \\ O & B' \end{bmatrix}$ be two triangular algebras over a commutative ring \mathcal{R} with $\frac{1}{2} \in \mathcal{R}$ and let $\mathfrak{l} : \mathcal{T} \rightarrow \mathcal{T}'$ be a Lie triple isomorphism. If*

- (1) *each centralizing trace of a bilinear mapping on \mathcal{T}' is proper,*
- (2) *at least one of A, B and at least one of A', B' are noncommutative,*
- (3) *M' is loyal,*

then $\mathfrak{l} = \pm \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m} : \mathcal{T} \rightarrow \mathcal{T}'$ is a Jordan homomorphism, \mathfrak{m} is one-to-one, and $\mathfrak{n} : \mathcal{T} \rightarrow \mathcal{Z}(\mathcal{T}')$ is a linear mapping vanishing on each second commutator. Moreover, if \mathcal{T}' is central over \mathcal{R} , then \mathfrak{m} is onto.

Proof. For arbitrary $x, z \in \mathcal{T}$, it is easy to see that \mathfrak{l} satisfies $[[\mathfrak{l}(x^2), \mathfrak{l}(x)], \mathfrak{l}(z)] = \mathfrak{l}([x^2, x], z) = 0$. Since \mathfrak{l} is onto, $[\mathfrak{l}(x^2), \mathfrak{l}(x)] \in \mathcal{Z}(\mathcal{T}')$ for all $x \in \mathcal{T}$. Replacing x by $\mathfrak{l}^{-1}(y)$, we get $[\mathfrak{l}(\mathfrak{l}^{-1}(y)^2), y] \in \mathcal{Z}(\mathcal{T}')$ for all $y \in \mathcal{T}'$. This means that the mapping $\mathfrak{T}_q(y) = \mathfrak{l}(\mathfrak{l}^{-1}(y)^2)$ is centralizing. Since \mathfrak{T}_q is also a trace of the bilinear mapping $\mathfrak{q} : \mathcal{T}' \times \mathcal{T}' \rightarrow \mathcal{T}'$, $\mathfrak{q}(y, z) = \mathfrak{l}(\mathfrak{l}^{-1}(y)\mathfrak{l}^{-1}(z))$, by the hypothesis (1) there exist $\lambda \in \mathcal{Z}(\mathcal{T}')$, a linear mapping $\mu_1 : \mathcal{T}' \rightarrow \mathcal{Z}(\mathcal{T}')$, and a trace $\nu_1 : \mathcal{T}' \rightarrow \mathcal{Z}(\mathcal{T}')$ of a bilinear mapping such that

$$\mathfrak{l}(\mathfrak{l}^{-1}(y)^2) = \lambda y^2 + \mu_1(y)y + \nu_1(y) \quad (4.1)$$

for all $y \in \mathcal{T}'$. Let $\mu = \mu_1 \mathfrak{l}$ and $\nu = \nu_1 \mathfrak{l}$. Then μ and ν are mappings of \mathcal{T} into $\mathcal{Z}(\mathcal{T}')$ and μ is linear. Hence (4.1) can be rewritten as

$$\mathfrak{l}(x^2) = \lambda \mathfrak{l}(x)^2 + \mu(x)\mathfrak{l}(x) + \nu(x) \quad (4.2)$$

for all $x \in \mathcal{T}$. We conclude that $\lambda \neq 0$. Otherwise, we have $\mathfrak{l}(x^2) - \mu(x)\mathfrak{l}(x) \in \mathcal{Z}(\mathcal{T}')$ by (4.2) and hence

$$\begin{aligned} \mathfrak{l}([x^2, y], [x, y]) &= [[\mathfrak{l}(x^2), \mathfrak{l}(y)], \mathfrak{l}([x, y])] \\ &= [[\mu(x)\mathfrak{l}(x), \mathfrak{l}(y)], \mathfrak{l}([x, y])] \\ &= \mu(x)[[\mathfrak{l}(x), \mathfrak{l}(y)], \mathfrak{l}([x, y])] \\ &= \mu(x)\mathfrak{l}([x, y], [x, y]) \\ &= 0 \end{aligned}$$

for all $x, y \in \mathcal{T}$. Consequently, $[x^2, y], [x, y] = 0$ for all $x, y \in \mathcal{T}$. According to our assumption this contradicts with [8, Lemma 2.7]. Thus $\lambda \neq 0$.

Now we define a linear mapping $\mathfrak{m} : \mathcal{T} \rightarrow \mathcal{T}'$ by

$$\mathfrak{m}(x) = \lambda \mathfrak{l}(x) + \frac{1}{2}\mu(x) \quad (4.3)$$

for the $x \in \mathcal{T}$. Of course, \mathbf{m} is a linear mapping. Our goal is to show that \mathbf{m} is a Jordan homomorphism. In view of (4.2) and (4.3), we have

$$\mathbf{m}(x^2) = \lambda \mathbf{l}(x^2) + \frac{1}{2} \mu(x) = \lambda^2 \mathbf{l}(x)^2 + \lambda \mu(x) \mathbf{l}(x) + \lambda \nu(x) + \frac{1}{2} \mu(x^2),$$

while

$$\mathbf{m}(x)^2 = (\lambda \mathbf{l}(x) + \frac{1}{2} \mu(x))^2 = \lambda^2 \mathbf{l}(x)^2 + \lambda \mu(x) \mathbf{l}(x) + \frac{1}{4} \mu(x)^2.$$

Comparing the above two identities we get

$$\mathbf{m}(x^2) - \mathbf{m}(x)^2 \in \mathcal{Z}(\mathcal{T}') \quad (4.4)$$

for all $x \in \mathcal{T}$. Linearizing (4.4) we obtain

$$\mathbf{m}(x \circ y) - \mathbf{m}(x) \circ \mathbf{m}(y) \in \mathcal{Z}(\mathcal{T}')$$

for all $x, y \in \mathcal{T}$. Define the mapping $\varepsilon : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{Z}(\mathcal{T}')$ by

$$\varepsilon(x, y) = \mathbf{m}(x \circ y) - \mathbf{m}(x) \circ \mathbf{m}(y). \quad (4.5)$$

Clearly, ε is a symmetric bilinear mapping. Of course, \mathbf{m} is a Jordan homomorphism if and only if $\varepsilon(x, y) = 0$ for all $x, y \in \mathcal{T}$. For any $x, y \in \mathcal{T}$, let us put $W = \mathbf{m}(x \circ (x \circ y))$. By (4.5) we have

$$\begin{aligned} W &= \mathbf{m}(x) \mathbf{m}(x \circ y) + \mathbf{m}(x \circ y) \mathbf{m}(x) + \varepsilon(x, x \circ y) \\ &= \mathbf{m}(x) \{ \mathbf{m}(x) \circ \mathbf{m}(y) + \varepsilon(x, y) \} + [\mathbf{m}(x) \circ \mathbf{m}(y) + \varepsilon(x, y)] \mathbf{m}(x) + \varepsilon(x, x \circ y) \\ &= \mathbf{m}(x)^2 \mathbf{m}(y) + 2 \mathbf{m}(x) \mathbf{m}(y) \mathbf{m}(x) + \mathbf{m}(y) \mathbf{m}(x)^2 + 2 \varepsilon(x, y) \mathbf{m}(x) + \varepsilon(x, x \circ y). \end{aligned}$$

On the other hand

$$\begin{aligned} W &= 2 \mathbf{m}(xyx) + \mathbf{m}(x^2 \circ y) \\ &= 2 \mathbf{m}(xyx) + \mathbf{m}(x^2) \circ \mathbf{m}(y) + \varepsilon(x^2, y) \\ &= 2 \mathbf{m}(xyx) + [\mathbf{m}(x^2) + \frac{1}{2} \varepsilon(x, x)] \mathbf{m}(y) \\ &\quad + \mathbf{m}(y) [\mathbf{m}(x^2) + \frac{1}{2} \varepsilon(x, x)] + \varepsilon(x^2, y) \\ &= 2 \mathbf{m}(xyx) + \mathbf{m}(x)^2 \mathbf{m}(y) + \mathbf{m}(y) \mathbf{m}(x)^2 \\ &\quad + \varepsilon(x, x) \mathbf{m}(y) + \varepsilon(x^2, y). \end{aligned}$$

Comparing the above two relations gives

$$\begin{aligned} \mathbf{m}(xyx) &= \mathbf{m}(x) \mathbf{m}(y) \mathbf{m}(x) + \varepsilon(x, y) \mathbf{m}(x) - \frac{1}{2} \varepsilon(x, x) \mathbf{m}(y) \\ &\quad + \frac{1}{2} \varepsilon(x, x \circ y) \mathbf{m}(y) - \frac{1}{2} \varepsilon(x^2, y). \end{aligned} \quad (4.6)$$

By completing linearization of (4.6) we obtain

$$\begin{aligned} \mathbf{m}(xyz + zyx) &= \mathbf{m}(x) \mathbf{m}(y) \mathbf{m}(z) + \mathbf{m}(z) \mathbf{m}(y) \mathbf{m}(x) + \varepsilon(x, y) \mathbf{m}(z) \\ &\quad + \varepsilon(z, y) \mathbf{m}(x) - \varepsilon(x, z) \mathbf{m}(y) + \frac{1}{2} \varepsilon(x, z \circ y) \\ &\quad + \frac{1}{2} \varepsilon(z, x \circ y) - \frac{1}{2} \varepsilon(x \circ z, y). \end{aligned} \quad (4.7)$$

Let us consider $U = \mathfrak{m}(xyx^2 + x^2yx)$. By (4.7) we know that

$$\begin{aligned} U &= \mathfrak{m}(x)\mathfrak{m}(y)\mathfrak{m}(x^2) + \mathfrak{m}(x^2)\mathfrak{m}(y)\mathfrak{m}(x) + \varepsilon(x, y)\mathfrak{m}(x^2) \\ &\quad + \varepsilon(x^2, y)\mathfrak{m}(x) - \varepsilon(x, x^2)\mathfrak{m}(y) + \frac{1}{2}\varepsilon(x, x^2 \circ y) \\ &\quad + \frac{1}{2}\varepsilon(x^2, x \circ y) - \frac{1}{2}\varepsilon(x^3, y). \end{aligned}$$

Since $\mathfrak{m}(x^2) = \mathfrak{m}(x)^2 + \frac{1}{2}\varepsilon(x, x)$, we get

$$\begin{aligned} U &= \mathfrak{m}(x)\mathfrak{m}(y)\mathfrak{m}(x)^2 + \mathfrak{m}(x)^2\mathfrak{m}(y)\mathfrak{m}(x) + \varepsilon(x, x)\mathfrak{m}(x)\mathfrak{m}(y) \\ &\quad + \frac{1}{2}\varepsilon(x, x)\mathfrak{m}(y)\mathfrak{m}(x) + \varepsilon(x, y)\mathfrak{m}(x)^2 + \varepsilon(x^2, y)\mathfrak{m}(x) \\ &\quad - \varepsilon(x, x^2)\mathfrak{m}(y) + \frac{1}{2}\varepsilon(x, y)\varepsilon(x, x) + \frac{1}{2}\varepsilon(x, x^2 \circ y) \\ &\quad + \frac{1}{2}\varepsilon(x^2, x \circ y) - \varepsilon(x^3, y). \end{aligned}$$

On the other hand, using (4.5) and (4.6) we have

$$\begin{aligned} U &= \mathfrak{m}((xyx) \circ x) \\ &= \mathfrak{m}(xyx) \circ \mathfrak{m}(x) + \varepsilon(xy, x) \\ &= \mathfrak{m}(x)\mathfrak{m}(y)\mathfrak{m}(x)^2 + \mathfrak{m}(x)^2\mathfrak{m}(y)\mathfrak{m}(x) + 2\varepsilon(x, y)\mathfrak{m}(x)^2 \\ &\quad - \frac{1}{2}\varepsilon(x, x)(\mathfrak{m}(y) \circ \mathfrak{m}(x)) + \varepsilon(x, x \circ y)\mathfrak{m}(x) - \varepsilon(x^2, y)\mathfrak{m}(x) + \varepsilon(xy, x). \end{aligned}$$

Comparing the above two relations yields

$$\begin{aligned} &\varepsilon(x, x)\mathfrak{m}(x) \circ \mathfrak{m}(y) - \varepsilon(x, y)\mathfrak{m}(x)^2 - \varepsilon(x, x^2)\mathfrak{m}(y) \\ &\quad + (2\varepsilon(x^2, y) - \varepsilon(x, x \circ y))\mathfrak{m}(x) \in \mathcal{Z}(\mathcal{T}') \end{aligned} \quad (4.8)$$

for all $x, y \in \mathcal{T}$. In particular, if $x = y$, we obtain

$$\varepsilon(x, x)\mathfrak{m}(x)^2 - \varepsilon(x, x^2)\mathfrak{m}(x) \in \mathcal{Z}(\mathcal{T}') \quad (4.9)$$

for all $x \in \mathcal{T}$. Therefore

$$\varepsilon(x, x)[[\mathfrak{m}(x)^2, u], [\mathfrak{m}(x), u]] = 0$$

for all $x \in \mathcal{T}, u \in \mathcal{T}'$, which can be in view of (4.3) rewritten as

$$\lambda^3 \varepsilon(x, x)[[\mathfrak{l}(x)^2, u], [\mathfrak{l}(x), u]] = 0.$$

We may assume that A' is noncommutative. Pick $a_1, a_2 \in A'$ such that $a_1[a_1, a_2]a_1 \neq 0$ (see the proof of [8, Lemma 2.7]). Setting

$$\mathfrak{l}(x_0) = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad u_0 = \begin{bmatrix} a_2 & m \\ 0 & 0 \end{bmatrix}$$

for some $x_0 \in \mathcal{T}$ and an arbitrary $m \in M'$ in the relation $\lambda^3 \varepsilon(x, x)[[\mathfrak{l}(x)^2, u], [\mathfrak{l}(x), u]] = 0$, we arrive at

$$\pi_{A'}(\lambda^3 \varepsilon(x_0, x_0))a_1[a_1, a_2]a_1m = 0$$

for all $m \in M'$. By the loyalty of M' it follows that $\pi_{A'}(\lambda^3 \varepsilon(x_0, x_0))a_1[a_1, a_2]a_1 = 0$. Hence $\pi_{A'}(\lambda^3 \varepsilon(x_0, x_0)) = 0$ by Lemma 3.2. This shows that $\lambda^3 \varepsilon(x_0, x_0) = 0$. Since $\lambda \neq 0$, $\varepsilon(x_0, x_0) = 0$ by Lemma 3.3. Taking $\varepsilon(x_0, x_0) = 0$ into (4.9) and then making the commutator with u_0 we can obtain $\varepsilon(x_0, x_0^2) = 0$. From (4.8) we get

$$\varepsilon(x_0, y)\mathfrak{m}(x_0)^2 + [-2\varepsilon(x_0^2, y) + \varepsilon(x_0, x_0 \circ y)]\mathfrak{m}(x_0) \in \mathcal{Z}(\mathcal{T}') \quad (4.10)$$

for all $y \in \mathcal{T}$. Let us commute the above relation with u_0 and then with $[\mathbf{m}(x_0), u_0]$ in order. We will eventually observe that $\varepsilon(x_0, y) = 0$ for all $y \in \mathcal{T}$. Then the equation (4.10) shows that $2\varepsilon(x_0^2, y)\mathbf{m}(x_0) \in \mathcal{Z}(\mathcal{T}')$ for all $y \in \mathcal{T}$. Therefore $\varepsilon(x_0^2, y)[\mathbf{m}(x_0), u_0] = 0$ and hence $\varepsilon(x_0^2, y) = 0$ for all $y \in \mathcal{T}$.

Next, we assert that $\varepsilon(x, y) = 0$. Substituting $x_0 + y$ for x by in (4.9) and using the fact $\varepsilon(x_0, y) = 0$, we have

$$\begin{aligned} & \varepsilon(y, y)\mathbf{m}(x_0)^2 + \varepsilon(y, y)\mathbf{m}(x_0) \circ \mathbf{m}(y) - \varepsilon(y, (x_0 + y)^2)\mathbf{m}(x_0) \\ & - \varepsilon(y, x_0 \circ y)\mathbf{m}(y) \in \mathcal{Z}(\mathcal{T}'). \end{aligned}$$

On the other hand, replacing x by $-x_0 + y$ in (4.9) we get

$$\begin{aligned} & \varepsilon(y, y)\mathbf{m}(x_0)^2 - \varepsilon(y, y)\mathbf{m}(x_0) \circ \mathbf{m}(y) + \varepsilon(y, (x_0 - y)^2)\mathbf{m}(x_0) \\ & + \varepsilon(y, x_0 \circ y)\mathbf{m}(y) \in \mathcal{Z}(\mathcal{T}'). \end{aligned}$$

Comparing the two relations it follows that

$$\varepsilon(y, y)\mathbf{m}(x_0)^2 - \varepsilon(y, x_0 \circ y)\mathbf{m}(x_0) \in \mathcal{Z}(\mathcal{T}').$$

Commuting with u_0 and then with $[\mathbf{m}(x_0), u_0]$, in view of (4.3) the above relation becomes

$$\varepsilon(y, y)[[\mathbf{l}(x_0)^2, u_0], [\mathbf{l}(x_0), u_0]] = 0. \quad (4.11)$$

Furthermore, $\varepsilon(y, y) = 0$ for all $y \in \mathcal{T}$. Hence $\varepsilon = 0$ by the symmetry of ε . This shows that \mathbf{m} is a Jordan homomorphism.

We claim that $\lambda = \pm 1$. By (4.3) it follows that

$$\begin{aligned} \lambda^2 \mathbf{m}([x, y], z] &= \lambda^3 \mathbf{l}([x, y], z] + \frac{1}{2} \lambda^2 \mu([x, y], z]) \\ &= [[\mathbf{m}(x), \mathbf{m}(y)], \mathbf{m}(z)] + \frac{1}{2} \lambda^2 \mu([x, y], z]) \end{aligned}$$

for all $x, y, z \in \mathcal{T}$. Moreover, we get

$$\lambda^2 \mathbf{m}([x, y], z] - [[\mathbf{m}(x), \mathbf{m}(y)], \mathbf{m}(z)] \in \mathcal{Z}(\mathcal{T}') \quad (4.12)$$

for all $x, y, z \in \mathcal{T}$. Considering (4.12) and using the facts $\mathbf{m}(x \circ y) = \mathbf{m}(x) \circ \mathbf{m}(y)$ and $[[x, y], z] = x \circ (y \circ z) - y \circ (x \circ z)$, we conclude that

$$(\lambda^2 - 1)[[\mathbf{m}(x), \mathbf{m}(y)], \mathbf{m}(z)] \in \mathcal{Z}(\mathcal{T}').$$

for all $x, y, z \in \mathcal{T}$. By (4.3) we know that $\lambda^3(\lambda^2 - 1)\mathbf{l}([x, y], z] \in \mathcal{Z}(\mathcal{T}')$. Since x, y, z are arbitrary elements in \mathcal{T} and \mathbf{l} is bijective, we eventually obtain $\lambda^3(\lambda^2 - 1) = 0$. Since $\lambda \neq 0$, we get $\lambda = \pm 1$.

Let us put $\mathbf{n}(x) = -\frac{1}{2}\mu(x)$. When $\lambda = 1$, then $\mathbf{l} = \mathbf{m} + \mathbf{n}$. It is easy to verify that $\mathbf{n}([x, y], z] = 0$ for all $x, y, z \in \mathcal{T}$. Note that \mathbf{m} is a Jordan homomorphism from \mathcal{T} into \mathcal{T}' and hence is a Lie triple homomorphism from \mathcal{T} into \mathcal{T}' . When $\lambda = -1$, then $\mathbf{n} = \mathbf{l} + \mathbf{m}$ is a Lie triple homomorphism from \mathcal{T} into $\mathcal{Z}(\mathcal{T}')$. Therefore $\mathbf{n}([x, y], z] = 0$ for all $x, y, z \in \mathcal{T}$.

We have to prove that \mathbf{m} is one-to-one. Suppose that $\mathbf{m}(w) = 0$ for some $w \in \mathcal{T}$. Then $\mathbf{l}(w) \in \mathcal{Z}(\mathcal{T}')$ and hence $w \in \mathcal{Z}(\mathcal{T})$. This implies that $\ker(\mathbf{m}) \subseteq \mathcal{Z}(\mathcal{T})$. That is, $\ker(\mathbf{m})$ is a Jordan ideal of $\mathcal{Z}(\mathcal{T})$. However, by Lemma 4.1 it follows that $\ker(\mathbf{m}) = 0$.

It remains to prove that \mathbf{m} is onto in case \mathcal{T}' is central over \mathcal{R} . Let us first show that $\mathbf{m}(1) = 1'$. Since \mathbf{l} is a Lie triple isomorphism, we have $\mathbf{l}(1) \in \mathcal{Z}(\mathcal{T}')$ and hence $\mathbf{m}(1) = \mathbf{l}(1) - \mathbf{n}(1) \in \mathcal{Z}(\mathcal{T}')$. Note that \mathbf{m} is a Jordan homomorphism. We

see that $2\mathbf{m}(x) = \mathbf{m}(x \circ 1) = 2\mathbf{m}(x)\mathbf{m}(1)$. Since $\frac{1}{2} \in \mathcal{R}$, $(\mathbf{m}(1) - 1')\mathbf{m}(x) = 0$, which can be rewritten as $(\mathbf{m}(1) - 1')\mathbf{l}(x) \in \mathcal{Z}(\mathcal{T}')$. Then $(\mathbf{m}(1) - 1')[\mathbf{l}(x), s] = 0$ for all $s \in \mathcal{T}'$. Therefore $(\mathbf{m}(1) - 1')[\mathcal{T}', \mathcal{T}'] = 0$. Consequently, $\pi_{A'}(\mathbf{m}(1) - 1')[A', A'] = 0$. This implies that $\pi_{A'}(\mathbf{m}(1) - 1') = 0$ and so $\mathbf{m}(1) = 1'$. Obviously, we may write $\mathbf{n}(x) = f(x)1'$ for some linear mapping $f: \mathcal{T} \rightarrow \mathcal{R}$. Since \mathbf{m} is \mathcal{R} -linear, we obtain that $\mathbf{l}(x) = \pm\mathbf{m}(x) + f(x)1' = \mathbf{m}(\pm x + f(x)1)$ for all $x \in \mathcal{T}$. Consequently \mathbf{m} is onto, since \mathbf{l} is bijective. The proof of the theorem is thus completed. \square

It would be helpful to point out that the proof just given in its first part is a modification of that of [11, Theorem 2] and we express it explicitly here for completeness. By a slight modification of this proof one could easily check the following proposition holds true.

Proposition 4.3. *Let \mathcal{T} and \mathcal{T}' be central unital algebras over a field F with $\text{char}(F) \neq 2$ and $\mathbf{l}: \mathcal{T} \rightarrow \mathcal{T}'$ be a Lie triple isomorphism. If*

- (1) *each centralizing trace of a bilinear mapping on \mathcal{T}' is proper,*
- (2) *\mathcal{T} and \mathcal{T}' do not satisfy the polynomial identity $[[x^2, y], [x, y]]$,*
- (3) *\mathcal{T}' does not satisfy the polynomial identity $[x, [y, w]]$,*

then $\mathbf{l} = \pm\mathbf{m} + \mathbf{n}$, where $\mathbf{m}: \mathcal{T} \rightarrow \mathcal{T}'$ is a Jordan isomorphism and $\mathbf{n}: \mathcal{T} \rightarrow \mathcal{Z}(\mathcal{T}')$ is a linear mapping vanishing on each second commutator.

We are in a position to state the main result of this section, which follows from Theorem 3.4 and Theorem 4.2.

Theorem 4.4. *Let $\mathcal{T} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ and $\mathcal{T}' = \begin{bmatrix} A' & M' \\ O & B' \end{bmatrix}$ be two triangular algebras over \mathcal{R} with $\frac{1}{2} \in \mathcal{R}$. Let $\mathbf{l}: \mathcal{T} \rightarrow \mathcal{T}'$ be a Lie triple isomorphism. If*

- (1) *each commuting linear mapping on A' or B' is proper,*
- (2) *$\pi_{A'}(\mathcal{Z}(\mathcal{T}')) = \mathcal{Z}(A') \neq A'$ and $\pi_{B'}(\mathcal{Z}(\mathcal{T}')) = \mathcal{Z}(B') \neq B'$,*
- (3) *either A or B is noncommutative,*
- (4) *M' is loyal,*

then $\mathbf{l} = \pm\mathbf{m} + \mathbf{n}$, where $\mathbf{m}: \mathcal{T} \rightarrow \mathcal{T}'$ is a Jordan homomorphism, \mathbf{m} is one-to-one, and $\mathbf{n}: \mathcal{T} \rightarrow \mathcal{Z}(\mathcal{T}')$ is a linear mapping vanishing on each second commutator. Moreover, if \mathcal{T}' is central over \mathcal{R} , then \mathbf{m} is onto.

Beidar, Brešar and Chebotar in [1] characterized Jordan isomorphisms of triangular matrix algebras over a connected commutative ring and obtained the following result. Let \mathcal{R} be a 2-torsionfree commutative ring with identity 1 and $\mathcal{T}_n(\mathcal{R})$ ($n \geq 2$) be the algebra of all upper triangular $n \times n$ ($n \geq 2$) matrices over \mathcal{R} . Then \mathcal{R} contains no idempotents except 0 and 1 (or equivalently, \mathcal{R} is a connected ring) if and only if every Jordan isomorphism of $\mathcal{T}_n(\mathcal{R})$ onto an arbitrary algebra over \mathcal{R} is either an isomorphism or an anti-isomorphism. Wong [48] extended the previous result by proving that if \mathcal{T} is a 2-torsion free unital indecomposable triangular algebra, then every Jordan isomorphism from \mathcal{T} onto another algebra is either an isomorphism or an anti-isomorphism.

Corollary 4.5. *Let \mathcal{R} be a commutative domain with $\frac{1}{2} \in \mathcal{R}$ and $\mathcal{T}_n(\mathcal{R})$ ($n \geq 2$) be the algebra of all $n \times n$ upper triangular matrices over \mathcal{R} . If $\mathbf{l}: \mathcal{T}_n(\mathcal{R}) \rightarrow \mathcal{T}_n(\mathcal{R})$ is a Lie triple isomorphism, then $\mathbf{l} = \pm\mathbf{m} + \mathbf{n}$, where $\mathbf{m}: \mathcal{T}_n(\mathcal{R}) \rightarrow \mathcal{T}_n(\mathcal{R})$ is an isomorphism or an anti-isomorphism and $\mathbf{n}: \mathcal{T}_n(\mathcal{R}) \rightarrow \mathcal{R}1$ is a linear mapping vanishing on each second commutator.*

Proof. Let us first consider the case of $n = 2$. Assume that $\mathfrak{l}: \mathcal{T}_2(\mathcal{R}) \longrightarrow \mathcal{T}_2(\mathcal{R})$ is a Lie triple isomorphism. Denote E_{ij} with $1 \leq i \leq j \leq 2$ as the usual matrix unit. Since $E_{12} = [E_{11}, [E_{11}, E_{12}]]$, we have $\mathfrak{l}(E_{12}) = rE_{12}$ for some invertible element $r \in \mathcal{R}^*$. Note that $[[\mathfrak{l}(I), \mathfrak{l}(X)], \mathfrak{l}(Y)] = 0$ for all $X, Y \in \mathcal{T}_2(\mathcal{R})$, which implies that $[\mathfrak{l}(I), \mathfrak{l}(X)] \in \mathcal{Z}(\mathcal{T}_2(\mathcal{R})) = \mathcal{R}I$. Hence $\mathfrak{l}(I) \in \mathcal{R}I$.

We assert that there exists a linear mapping \mathfrak{g} from the diagonal subalgebra \mathcal{D}_2 into itself and a scalar $s \in \mathcal{R}$ such that

$$\mathfrak{l}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \mathfrak{g}\left(\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\right) + \begin{bmatrix} 0 & rb + s(a - c) \\ 0 & 0 \end{bmatrix}$$

for all $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \mathcal{T}_2(\mathcal{R})$. In fact, we know that for arbitrary $\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \in \mathcal{T}_2(\mathcal{R})$, there exist \mathcal{R} -linear mappings $\mathfrak{f}_i, \mathfrak{g}_i, \mathfrak{h}_i: \mathcal{R} \longrightarrow \mathcal{R}$ ($i = 1, 2, 3$) such that

$$\mathfrak{l}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{pmatrix} \mathfrak{f}_1(a) + \mathfrak{f}_2(b) + \mathfrak{f}_3(c) & \mathfrak{g}_1(a) + \mathfrak{g}_2(b) + \mathfrak{g}_3(c) \\ 0 & \mathfrak{h}_1(a) + \mathfrak{h}_2(b) + \mathfrak{h}_3(c) \end{pmatrix}.$$

Since $\mathfrak{l}(E_{12}) = rE_{12}$, we have $\mathfrak{f}_2(b) = 0 = \mathfrak{h}_2(b)$. Thus

$$\mathfrak{l}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{pmatrix} \mathfrak{f}_1(a) + \mathfrak{f}_3(c) & rb + \mathfrak{g}_1(a) + \mathfrak{g}_3(c) \\ 0 & \mathfrak{h}_1(a) + \mathfrak{h}_3(c) \end{pmatrix}.$$

Note that $\mathfrak{g}_1(a) = a\mathfrak{g}_1(1)$ and $\mathfrak{g}_3(c) = c\mathfrak{g}_3(1)$. On the other hand, it follows from the fact $\mathfrak{l}(I) \in \mathcal{R}I$ that $\mathfrak{g}_1(1) + \mathfrak{g}_3(1) = 0$. Let $\mathfrak{g}_1(1) = s$ and then the above arguments imply our assertion. Let us write $S = \begin{bmatrix} r & -s \\ 0 & 1 \end{bmatrix}$. Then $S^{-1} = \begin{bmatrix} r^{-1} & sr^{-1} \\ 0 & 1 \end{bmatrix}$. We define $\mathfrak{j}(T) = \mathfrak{l}(S^{-1}TS)$ for all $T \in \mathcal{T}_2(\mathcal{R})$. Then \mathfrak{j} is a Lie triple isomorphism from $\mathcal{T}_2(\mathcal{R})$ into itself and

$$\mathfrak{j}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \mathfrak{g}\left(\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\right) + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

This implies that $\mathfrak{j}(E_{12}) = E_{12}$, $\mathfrak{j}|_{\mathcal{D}_2} = \mathfrak{g}$. Note that \mathfrak{j} is obtained by \mathfrak{l} composed with an inner automorphism. Therefore we only to prove the triple isomorphism \mathfrak{j} is of the standard form. Suppose that $\mathfrak{j}(E_{11}) = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$. By $E_{12} = [\mathfrak{j}(E_{11}), \mathfrak{j}(E_{11}), E_{12}]$ it follows that $(x - y)^2 = 1$. Since \mathcal{R} is a domain, we obtain $x - y = \pm 1$.

Case 1. If $x - y = 1$, then $\mathfrak{j}(E_{11}) = E_{11} + yI$ and $\mathfrak{j}(E_{22}) = \mathfrak{j}(I) - \mathfrak{j}(E_{11}) = E_{22} + zI$ for some $z \in \mathcal{R}$. It is easy to verify that $\det(\mathfrak{j}) = 1 + y + z \in \mathcal{R}^*$ as \mathfrak{j} is bijective. In view of [23, Page 103], \mathfrak{j} is of the standard form.

Case 2. When $x - y = -1$, note that $-\mathfrak{j}$ is also a triple isomorphism. Define $\mathfrak{t}(T) = -\mathfrak{j}(U^{-1}TU)$ for all $T \in \mathcal{T}_2(\mathcal{R})$, where $U = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\mathfrak{t}\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \mathfrak{g}\left(\begin{bmatrix} -a & 0 \\ 0 & -c \end{bmatrix}\right) + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

This implies that $\mathfrak{t}(E_{12}) = E_{12}$, $\mathfrak{t}|_{\mathcal{D}_2} = -\mathfrak{g}$. Moreover, $\mathfrak{t}(E_{11}) = -\mathfrak{j}(E_{11}) = \begin{bmatrix} -x & 0 \\ 0 & -y \end{bmatrix}$. This means that \mathfrak{t} satisfies the assumption of Case 1. Therefore \mathfrak{t} and hence \mathfrak{j} is of the standard form.

Suppose that $n > 2$. We may write

$$\mathcal{T} = \begin{bmatrix} \mathcal{R} & M_{1 \times (n-1)}(\mathcal{R}) \\ O & \mathcal{T}_{n-1}(\mathcal{R}) \end{bmatrix}$$

By Corollary 3.12 each centralizing trace of a bilinear mapping on $\mathcal{T}_n(\mathcal{R})$ is proper. Moreover, $\mathcal{T}_n(\mathcal{R})$ is commutative and $M_{1 \times (n-1)}(\mathcal{R})$ is a loyal $(\mathcal{R}, \mathcal{T}_{n-1}(\mathcal{R}))$ -bimodule. Thus the assumptions (1)–(3) of Theorem 4.2 hold in this case. Applying Theorem 4.2 and [48, Theorem 3.2] yields the conclusion. \square

In view of Proposition 4.3 and [48, Theorem 3.3] we can show

Corollary 4.6. *Let \mathcal{N} and \mathcal{N}' be nests on a Hilbert space H , $\mathcal{Alg}(\mathcal{N})$ and $\mathcal{Alg}(\mathcal{N}')$ be the nest algebras associated with \mathcal{N} and \mathcal{N}' , respectively. If $\mathfrak{l}: \mathcal{Alg}(\mathcal{N}) \rightarrow \mathcal{Alg}(\mathcal{N}')$ is a Lie triple isomorphism, then $\mathfrak{l} = \pm \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}: \mathcal{Alg}(\mathcal{N}) \rightarrow \mathcal{Alg}(\mathcal{N}')$ is an isomorphism or an anti-isomorphism and $\mathfrak{n}: \mathcal{Alg}(\mathcal{N}) \rightarrow \mathbb{C}1'$ is a linear mapping vanishing on each second commutator.*

Proof. Note that the corollary trivially holds in case $\dim_{\mathbb{C}} H = 1$ (namely, $\mathfrak{l} = \text{id} + (\mathfrak{l} - \text{id})$). If $\dim_{\mathbb{C}} H = 2$, we have either $\mathcal{Alg}(\mathcal{N}) = \mathcal{Alg}(\mathcal{N}') \cong \mathcal{T}_2(\mathbb{C})$ or $\mathcal{Alg}(\mathcal{N}) = \mathcal{Alg}(\mathcal{N}') \cong \mathcal{M}_2(\mathbb{C})$. Corollary 4.5 implies the first case, while the second case follows from [18, Theorem 3.5].

Suppose that $\dim_{\mathbb{C}} H > 2$. Then each nest algebra is central over \mathbb{C} . We assert that the conditions (1)-(3) of Proposition 4.3 are satisfied in this case. The condition (1) is due to Corollary 3.13. While (2) and (3) are due to [8, Remark 2.13]. Applying Proposition 4.3 and [48, Theorem 3.3] yields the desired result. \square

5. TOPICS FOR FURTHER RESEARCH

Although the main purpose of the current article is to study centralizing traces and Lie triple isomorphisms of triangular algebras, the structure of centralizing traces and Lie triple isomorphisms of other associative algebras also has a great interest and draw more people's our attention. In this section we will present several potential topics for future further research.

Let us begin with the definition of generalized matrix algebras given by a Morita context. Let \mathcal{R} be a commutative ring with identity. A *Morita context* consists of two \mathcal{R} -algebras A and B , two bimodules ${}_A M_B$ and ${}_B N_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN}: M \otimes_B N \rightarrow A$ and $\Psi_{NM}: N \otimes_A M \rightarrow B$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} M \otimes_B N \otimes_A M & \xrightarrow{\Phi_{MN} \otimes I_M} & A \otimes M \\ \downarrow I_M \otimes \Psi_{NM} & & \downarrow \cong \\ M \otimes_B B & \xrightarrow{\cong} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} N \otimes_A M \otimes_B N & \xrightarrow{\Psi_{NM} \otimes I_N} & B \otimes N \\ \downarrow I_N \otimes \Phi_{MN} & & \downarrow \cong \\ N \otimes_A A & \xrightarrow{\cong} & N. \end{array}$$

Let us write this Morita context as $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$. We refer the reader to [40] for the basic properties of Morita contexts. If $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$\left[\begin{array}{cc} A & M \\ N & B \end{array} \right] = \left\{ \left[\begin{array}{cc} a & m \\ n & b \end{array} \right] \middle| a \in A, m \in M, n \in N, b \in B \right\}$$

form an \mathcal{R} -algebra under matrix-like addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. Such an \mathcal{R} -algebra is usually called a *generalized matrix algebra* of order 2 and is denoted by

$$\mathcal{G} = \left[\begin{array}{cc} A & M \\ N & B \end{array} \right].$$

In a similar way, one can define a generalized matrix algebra of order $n > 2$. It was shown that up to isomorphism, arbitrary generalized matrix algebra of order n ($n \geq 2$) is a generalized matrix algebra of order 2 [29, Example 2.2]. If one of the modules M and N is zero, then \mathcal{G} exactly degenerates to an *upper triangular algebra* or a *lower triangular algebra*. In this case, we denote the resulted upper triangular algebra (resp. lower triangular algebra) by

$$\mathcal{T}^u = \begin{bmatrix} A & M \\ O & B \end{bmatrix} \quad \left(\text{resp. } \mathcal{T}^l = \begin{bmatrix} A & O \\ N & B \end{bmatrix} \right)$$

Let $\mathcal{M}_n(\mathcal{R})$ be the full matrix algebra consisting of all $n \times n$ matrices over \mathcal{R} . It is worth to point out that the notion of generalized matrix algebras efficiently unifies triangular algebras with full matrix algebras together. The distinguished feature of our systematic work is that we deal all questions related to (non-)linear mappings of triangular algebras and full matrix algebras under a unified frame, which is the admired generalized matrix algebras frame, see [25, 29, 30, 47, 49, 50].

Let $\mathcal{T} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ be a 2-torsion free triangular algebra over commutative ring \mathcal{R} and $\mathfrak{q}: \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ be an \mathcal{R} -bilinear mapping. Theorem 3.4 shows that under some mild conditions, every centralizing trace $\mathfrak{T}_{\mathfrak{q}}: \mathcal{T} \rightarrow \mathcal{T}$ of \mathfrak{q} has the proper form. As you see in the proof of this theorem, one of the most key steps is that every centralizing trace $\mathfrak{T}_{\mathfrak{q}}: \mathcal{T} \rightarrow \mathcal{T}$ of \mathfrak{q} is commuting. Brešar in [10] proved that in certain rings, in particular, prime rings of characteristic different from 2 and 3, every centralizing trace of arbitrary bilinear mapping is commuting. It is natural to ask the following question

Question 5.1. Let $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ be a generalized matrix algebra over \mathcal{R} and $\mathfrak{q}: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ be an \mathcal{R} -bilinear mapping. Under what conditions, every centralizing trace $\mathfrak{G}_{\mathfrak{q}}: \mathcal{G} \rightarrow \mathcal{G}$ of \mathfrak{q} has the proper form?

Calderón Martín and Martín González in [18] gave a characterization of Lie triple automorphisms of full matrix algebras over complex field \mathbb{C} . Let $\mathfrak{l}: \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})$ ($n > 1$) be a Lie triple automorphism. Then there exists an automorphism, an anti-automorphism, the negative of an automorphism or the negative of an anti-automorphism $\mathfrak{m}: \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})$ such that $\mathfrak{n} = \mathfrak{l} - \mathfrak{m}$ is a linear mapping from $\mathcal{M}_n(\mathbb{C})$ onto its center sending all second commutators to zero. In light of this result and our Theorem 4.4 we propose

Conjecture 5.2. Let $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ and $\mathcal{G}' = \begin{bmatrix} A' & M' \\ N' & B' \end{bmatrix}$ be generalized matrix algebras over \mathcal{R} with $1/2 \in \mathcal{R}$. Let $\mathfrak{l}: \mathcal{G} \rightarrow \mathcal{G}'$ be a Lie triple isomorphism. If

- (1) each commuting linear mapping on A' or B' is proper,
- (2) $\pi_{A'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(A') \neq A'$ and $\pi_{B'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(B') \neq B'$,
- (3) either A or B is noncommutative,
- (4) M' is loyal,

then $\mathfrak{l} = \pm \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}: \mathcal{G} \rightarrow \mathcal{G}'$ is a Jordan homomorphism, \mathfrak{m} is one-to-one, and $\mathfrak{n}: \mathcal{G} \rightarrow \mathcal{Z}(\mathcal{G}')$ is a linear mapping vanishing on each second commutator. Moreover, if \mathcal{G}' is central over \mathcal{R} , then \mathfrak{m} is surjective.

More recently, some researchers extend the result about Lie isomorphisms between nest algebras on Hilbert spaces by Marcoux and Sourour [34] to the Banach space case, see [43] and [46]. Therefore it is deserved to pay much more attention to centralizing traces and Lie triple isomorphisms of nest algebras on Banach spaces.

Basing on Corollary 3.13 we have the following question.

Question 5.3. Let X be a Banach space, \mathcal{N} be a nest of X and $\mathcal{Alg}(\mathcal{N})$ be the nest algebra associated with \mathcal{N} . Suppose that $\mathfrak{q}: \mathcal{Alg}(\mathcal{N}) \times \mathcal{Alg}(\mathcal{N}) \rightarrow \mathcal{Alg}(\mathcal{N})$ is an \mathcal{R} -bilinear mapping. Then every centralizing trace $\mathfrak{T}_{\mathfrak{q}}: \mathcal{Alg}(\mathcal{N}) \rightarrow \mathcal{Alg}(\mathcal{N})$ of \mathfrak{q} is proper.

Furthermore, similiar to Corollary 4.6 we conjecture

Conjecture 5.4. Let \mathcal{N} and \mathcal{N}' be nests on a Banach space X , $\mathcal{Alg}(\mathcal{N})$ and $\mathcal{Alg}(\mathcal{N}')$ be the nest algebras associated with \mathcal{N} and \mathcal{N}' , respectively. If $\mathfrak{l}: \mathcal{Alg}(\mathcal{N}) \rightarrow \mathcal{Alg}(\mathcal{N}')$ is a Lie triple isomorphism, then $\mathfrak{l} = \pm \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}: \mathcal{Alg}(\mathcal{N}) \rightarrow \mathcal{Alg}(\mathcal{N}')$ is an isomorphism or an anti-isomorphism and $\mathfrak{n}: \mathcal{Alg}(\mathcal{N}) \rightarrow \mathbb{C}1'$ is a linear mapping vanishing on each second commutator.

REFERENCES

- [1] K. I. Beidar, M. Brešar and M. A. Chebotar, *Jordan isomorphisms of triangular matrix algebras over a connected commutative ring*, Linear Algebra Appl., **312** (2000), 197-201.
- [2] K. I. Beidar, M. Brešar and M. A. Chebotar, *Functional identities on upper triangular matrix algebras*, J. Math. Sci., **102** (2000), 4557-4565.
- [3] K. I. Beidar, M. Brešar, M. A. Chebotar and W. S. Martindale 3rd, *On Herstein's Lie map conjectures, III*, J. Algebra, **249** (2002), 59-94.
- [4] D. Benkovič, *Lie derivations on triangular matrices*, Linear Multilinear Algebra, **55** (2007), 619-626.
- [5] D. Benkovič, *Biderivations of triangular algebras*, Linear Algebra Appl., **431** (2009), 1587-1602.
- [6] D. Benkovič, *Generalized Lie derivations on triangular algebras*, Linear Algebra Appl., **434** (2011), 1532-1544.
- [7] D. Benkovič, *Lie triple derivations on triangular matrices*, Algebra Colloq., **18** (2011), Special Issue No.1, 819-826.
- [8] D. Benkovič and D. Eremita, *Commuting traces and commutativity preserving maps on triangular algebras*, J. Algebra, **280** (2004), 797-824.
- [9] D. Benkovič and D. Eremita, *Multiplicative Lie n -derivations of triangular rings*, Linear Algebra Appl., **436** (2012), 4223-4240.
- [10] M. Brešar, *On a generalization of the notion of centralizing mappings*, Proc. Amer. Math. Soc., **114** (1992), 641-649.
- [11] M. Brešar, *Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings*, Trans. Amer. Math. Soc., **335** (1993), 525-546.
- [12] M. Brešar, *Centralizing mappings and derivations in prime rings*, J. Algebra, **156** (1993), 385-394.
- [13] M. Brešar, *Commuting maps: a survey*, Taiwanese J. Math., **8** (2004), 361-397.
- [14] M. Brešar and D. Eremita and A. R. Villena, *Functional identities in Jordan algebras: associating traces and Lie triple isomorphisms*, Comm. Algebra, **31** (2003), 1207-1234.
- [15] M. Brešar and P. Šemrl, *Commuting traces of biadditive maps revisited*, Comm. Algebra, **31** (2003), 381-388.
- [16] A. J. Calderón Martín and C. Martín González, *Lie isomorphisms on H^* -algebras*, Comm. Algebra, **31** (2003), 323-333.
- [17] A. J. Calderón Martín and C. Martín González, *The Banach-Lie group of Lie triple automorphisms of an H^* -algebra*, Acta Math. Sci. (Ser. English), **30** (2010), 1219-1226.
- [18] A. J. Calderón Martín and C. Martín González, *A linear approach to Lie triple automorphisms of H^* -algebras*, J. Korean Math. Soc., **48** (2011), 117-132.
- [19] A. J. Calderón Martín and M. Haralampidou, *Lie mappings on locally m -convex H^* -algebras*, International Conference on Topological Algebras and their Applications. ICTAA 2008, 42-51, Math. Stud. (Tartu), 4, Est. Math. Soc., Tartu, 2008.
- [20] W. S. Cheung, *Maps on triangular algebras*, Ph.D. Dissertation, University of Victoria, 2000. 172pp.

- [21] W. S. Cheung, *Commuting maps of triangular algebras*, J. London Math. Soc., **63** (2001), 117-127.
- [22] W. S. Cheung, *Lie derivations of triangular algebras*, Linear Multilinear Algebra, **51** (2003), 299-310.
- [23] D. Ž. Doković, *Automorphisms of the Lie algebra of upper triangular matrices over a connected commutative ring*, J. Algebra, **170** (1994), 101-110.
- [24] G. Dolinar, *Maps on M_n preserving Lie products*, Publ. Math. Debrecen, **71** (2007), 467-477.
- [25] Y.-Q. Du and Y. Wang, *Lie derivations of generalized matrix algebras*, Linear Algebra Appl., **437** (2012), 2719-2726.
- [26] I. N. Herstein, *Lie and Jordan structures in simple, associative rings*, Bull. Amer. Math. Soc., **67** (1961), 517-531.
- [27] L. Hua, *A theorem on matrices over an s -field and its applications*, J. Chinese Math. Soc., (N.S.) **1** (1951), 110-163.
- [28] P. -H. Lee, T. -L. Wong, J. -S. Lin and R. -J. Wang, *Commuting traces of multiadditive mappings*, J. Algebra, **193** (1997), 709-723.
- [29] Y.-B. Li and F. Wei, *Semi-centralizing maps of generalized matrix algebras*, Linear Algebra Appl., **436** (2012), 1122-1153.
- [30] Y.-B. Li, L. van Wyk and F. Wei, *Jordan derivations and antiderivations of generalized matrix algebras*, Oper. Matrices, **7** (2013), 399-415.
- [31] F.-Y. Lu, *Jordan isomorphisms of nest algebras*, Proc. Amer. Math. Soc., **131** (2003), 147-154.
- [32] F.-Y. Lu, *Lie isomorphisms of reflexive algebras*, J. Funct. Anal., **240** (2006), 84-104.
- [33] L. W. Marcoux and A. R. Sourour, *Commutativity preserving maps and Lie automorphisms of triangular matrix algebras*, Linear Algebra Appl., **288** (1999), 89-104.
- [34] L. W. Marcoux and A. R. Sourour, *Lie isomorphisms of nest algebras*, J. Funct. Anal., **164** (1999), 163-180.
- [35] M. Mathieu, *Lie mappings of C^* -algebras*, Nonassociative algebra and its applications, 229-234, Lecture Notes in Pure and Appl. Math., **211**, Dekker, New York, 2000.
- [36] J. H. Mayne, *Centralizing automorphisms of prime rings*, Canad. Math. Bull., **19** (1976), 113-115.
- [37] C. R. Miers, *Lie isomorphisms of factors*, Trans. Amer. Math. Soc., **147** (1970), 55-63.
- [38] C. R. Miers, *Lie homomorphism of operator algebras*, Pacific J. Math., **38** (1971), 717-735.
- [39] C. R. Miers, *Lie $*$ -triple homomorphisms into von Neumann algebras*, Proc. Amer. Math. Soc., **58** (1976), 169-172.
- [40] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Diagaku Sect. A, **6** (1958), 83-142.
- [41] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1093-1100.
- [42] X.-F. Qi and J.-C. Hou, *Characterization of ξ -Lie multiplicative isomorphisms*, Oper. Matrices, **4** (2010), 417-429.
- [43] X.-F. Qi and J.-C. Hou, *Characterization of Lie multiplicative isomorphisms between nest algebras*, Sci. China Math., **54** (2011), 2453-2462.
- [44] P. Šemrl, *Non-linear commutativity preserving maps*, Acta Sci. Math. (Szeged), **71** (2005), 781-819.
- [45] A. R. Sourour, *Maps on triangular matrix algebras*, Problems in applied mathematics and computational intelligence, 92-96, Math. Comput. Sci. Eng., World Sci. Eng. Soc. Press, Athens, 2001.
- [46] T. Wang and F.-Y. Lu, *Lie isomorphisms of nest algebras on Banach spaces*, J. Math. Anal. Appl., **391** (2012), 582-594.
- [47] Y. Wang and Y. Wang, *Multiplicative Lie n -derivations of generalized matrix algebras*, Linear Algebra Appl., In Press, <http://dx.doi.org/10.1016/j.laa.2012.10.052>.
- [48] T.-L. Wong, *Jordan isomorphisms of triangular rings*, Proc. Amer. Math. Soc., **133** (2005), 3381-3388.
- [49] Z.-K. Xiao and F. Wei, *Commuting mappings of generalized matrix algebras*, Linear Algebra Appl., **433** (2010), 2178-2197.
- [50] Z.-K. Xiao and F. Wei, *Commuting traces and Lie isomorphisms on generalized matrix algebras*, Submitted to Sci. China Math.
- [51] X.-P. Yu and F.-Y. Lu, *Maps preserving Lie product on $B(X)$* , Taiwanese J. Math., **12** (2008), 793-806.

- [52] J.-H. Zhang and F.-J. Zhang, *Nonlinear maps preserving Lie products on factor von Neumann algebras*, Linear Algebra Appl., **429** (2008), 18-30.

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